International Journal of Theoretical Physics, Vol. 40, No. 1, 2001

Pseudotwistors[1]

Julian Ławrynowicz[2] and Osamu Suzuki[3]

Received February 15, 2000; revised July 5, 2000

We deal with the Hermitian Hurwitz pairs of signature (σ, s), $\sigma + s = 5 + 4\mu$, $|\sigma + 1 - s| = 2 + 4m$;$\mu$, $m = 0, 1, \ldots$. We introduce the *Hurwitz twistors* for signature (3, 2) and its dual (1, 4) and we indicate the relationship between Hurwitz and Penrose twistors. The signatures (1, 8) and (7, 6) give rise to *pseudotwistors* and *bitwistors*, respectively. For pseudotwistors, we prove a counterpart of the Penrose theorem in the local version, on real analytic solutions of the related spinor equations versus harmonic forms, and in the semiglobal version, on holomorphic solutions of those equations versus Dolbeault cohomology groups. We prove an atomization theorem: There exist complex structures on isometric embeddings for the Hermitian Hurwitz pairs so that the embeddings are real parts of holomorphic mappings.

1. INTRODUCTION

Penrose (1977) observed that the points of the Minkowski space-time can be represented by two-dimensional linear subspaces of a four-dimensional $\mathbb{C}$-space with a Hermitian form of signature $(++--)$. He called this a flat twistor space, and the deformation of complex structures yielded the twistor program.

Ławrynowicz and Rembieliński (1985, 1986a, b, 1987) initiated a geometrization of the Hurwitz problem of the composition of quadratic forms and a geometrical study of the related differential operators of the Cauchy–Riemann, Dirac, and Fueter types by introducing the Hurwitz pairs, also in

[1] The research of J. Ł. was supported by the State Committee for Scientific Research (KBN) grant PB 2 PO3A 010 17 and by grants of the University of Łódź No. 505/626 and 252.
[2] Polska Akademia Nauk, Instytut Matematyczny, ulica Banacha 22, PL-90-238 Łódź; Uniwersytet Lódzki, Katedra Fizyki Ciala Stalego, ulica Pomorska 149/153, PL-90-236 Łódź, Poland; e-mail: jlawryno@krysia.uni.lodz.pl
[3] Nihon University, College of Humanities and Sciences, Department of Mathematics, Sakurajosui 3-25-40, Setagaya-ku, Tokyo 156, Japan; e-mail: osuzuki@am.chs.nihon-u.ac.jp

0020-7748/01/0100-0387$19.50/0

the hyperbolic case discussed by Penrose (1977) and Wells (1982). Ławrynowicz and Suzuki (1998) proved counterparts of two Penrose theorems within the Hurwitz pairs.

Let $\mathbb{C}^n(\kappa)$ be an n-dimensional $\mathbb{C}$-space with Hermitian metric of signature (ρ, r), $\rho + r = n$,

$$\kappa \equiv I_{\rho,r} := \begin{pmatrix} I_\rho & 0 \\ 0 & -I_r \end{pmatrix}, \qquad f, g \in \mathbb{C}^n, \quad \langle\langle f, g\rangle\rangle_\kappa := f^*\kappa g$$

Let $\mathbb{R}^p(\eta)$ be a p-dimensional $\mathbb{R}$-space with the symmetric metric of signature (σ, s), $\sigma + s = p$:

$$\eta \equiv I_{\sigma,s} := \begin{pmatrix} I_\sigma & 0 \\ 0 & -I_s \end{pmatrix}, \qquad a, b \in \mathbb{R}^p, \quad \langle a, b\rangle_\eta := a^T\eta b$$

Definition 1.1 (Hermitian Hurwitz pair). Let $H \equiv (\mathbb{C}^n(\kappa), \mathbb{R}^p(\eta))$ with $s \neq 0$. For $s = 0$ the situation is different and easier. If there exists a mapping $\circ$: $\mathbb{R}^p(\eta) \otimes_{\mathbb{R}} \mathbb{C}^n(\kappa) \to \mathbb{C}^n(\kappa)$ such that, for $f \in \mathbb{C}^n(\kappa)$ and $a \in \mathbb{R}^p(\eta)$, we have $\langle a, a\rangle_\eta\langle\langle f, f\rangle\rangle_\kappa = \langle\langle a \circ f, a \circ f\rangle\rangle_\kappa$ and, moreover, H is *irreducible*, i.e., there exists no subspace $\{0\} \subsetneq V \subsetneq \mathbb{C}^n$ such that $\mathrm{im}(\circ|\mathbb{R}^p(\eta) \otimes V) \subset V$, then H is called a *Hermitian Hurwitz pair.*

Definition 1.2 (Hurwitz algebra). A central Clifford algebra of order $(\sigma - 1, s)$, $\sigma + s = p$, which admits a representation $(S_1, \ldots, S_p)$ with the condition $S_\alpha^\# \equiv \kappa S_\alpha^* \kappa^{-1} = S_\alpha$, $\alpha \neq t$, t fixed, is called a *Hurwitz algebra* and denoted by $\mathscr{H}_{\sigma-1,s}$.

Hurwitz algebra defines in $\mathbb{R}^p(\kappa)$ differential operators of the Dirac–Fueter type,

$$D^\delta_{\sigma,s}\,\Psi = 0, \qquad D^\delta_{\sigma,s} := \sum_{\alpha\neq t} iS_\alpha\,\partial^\alpha + \delta I_n\,\partial^t, \qquad \delta = 1 \text{ or } 0 \tag{1}$$

The name *generalized Fueter equation* is used for $\delta = 1$ and *generalized Dirac equation* for $\delta = 0$. Let $S^{\mathrm{F}}_{\sigma-1,s}(H)$ and $S^{\mathrm{D}}_{\sigma-1,s}(H)$ denote the linear space of solutions of Eq. (1) in an open set U for $\delta = 1$ and $\delta = 0$, respectively.

In the sequel, $\{\epsilon_\alpha\}$ denotes the basis of $\mathbb{R}^{k+l}(I_{k,l})$, $\{e_j\}$ denotes the basis of $\mathbb{C}^{p+q}(I_{p,q})$, and

$$\circ: \quad \mathbb{R}^{k+l}(I_{k,l}) \otimes_{\mathbb{R}} \mathbb{C}^{p+q}(I_{p,q}) \to \mathbb{C}^{p+q}(I_{p,q})$$

$$\epsilon_\alpha \circ e_k = \sum_{j=1,\ldots,p+q} C^j_{\alpha k}e_j, \qquad C_\alpha \equiv (C^j_{\alpha k}), \qquad C^\#_\alpha \equiv \kappa C^*_\alpha\,\kappa^{-1}$$

$$C_\alpha \circ C_\beta + C_\beta \circ C_\alpha = \eta_{\alpha,\beta}$$

We define a matrix $\mathbb{C}$-algebra $\mathscr{A}_{k,l} \equiv \mathrm{gen}\,\{C^\#_\alpha C_\beta: \alpha \leq \beta\} \subset \mathrm{End}\;\mathbb{C}^{p+q}$. Every $x \in \mathscr{A}_{k,l}$ can be written uniquely as

$$x = \sum_{k=0}^{k+1-1} x_k, \qquad x_k = \sum_{\alpha_1<\beta_1<\cdots<\alpha_k<\beta_k} \xi_{\alpha_1\beta_1\cdots\alpha_k\beta_k} C^{\#}_{\alpha_1} C_{\beta_1} \ldots C^{\#}_{\alpha_k} C_{\beta_k}$$

with $x_0 = \xi_0 I_{p+q}$ and im $x := x - x_0$

Definition 1.3 (Hurwitz twistor; Ławrynowicz and Suzuki, 1998a). An element $x \in \mathcal{A}_{2,3} \subset \text{End}\, \mathbb{C}^{2+2}$ is called a *Hurwitz twistor* whenever

$$x = \sum_{\alpha<\beta} \xi_{\alpha\beta} C^{\#}_{\alpha} C_{\beta} \qquad \text{and} \qquad \text{im}\, x^2 = 0 \tag{2}$$

$$P^1 = \mathcal{I}_H := \{x = \sum_{\alpha<\beta} \xi_{\alpha\beta} C^{\#}_{\alpha} C_{\beta}\colon \text{im}\, x^2 = 0\} \tag{3}$$

Theorems A and B below motivate the name 'Hurwitz twistor'.

Lemma A. The expression (2) is an element of $\mathcal{I}_H = P^1$ if and only if the following $\binom{5}{4}$ equations hold:

$$\xi_{12}\xi_{45} - \xi_{14}\xi_{25} + \xi_{15}\xi_{24} = 0$$
$$\xi_{12}\xi_{34} - \xi_{13}\xi_{24} + \xi_{14}\xi_{23} = 0$$
$$\xi_{13}\xi_{45} - \xi_{14}\xi_{35} + \xi_{15}\xi_{34} = 0$$
$$\xi_{12}\xi_{35} - \xi_{13}\xi_{25} + \xi_{15}\xi_{23} = 0$$
$$\xi_{23}\xi_{45} - \xi_{24}\xi_{35} + \xi_{25}\xi_{34} = 0$$

Proof. The calculation of im $x^2 = 0$. ■

Lemma B. $\mathcal{I}_H = P^1$ admits the structure of a flag manifold:

$$\mathcal{I}_H = \left\{ (L^1_1, L^1_2)\colon \begin{array}{l} L^1_1, L^1_2 \text{ are linear subspaces of } \mathbb{C}^4 \\ L^1_1 \subset L^1_2, \dim L^1_1 = 1, \dim L^1_2 = 2 \end{array} \right\}$$

Hence $\mathcal{I}_H$ becomes a complex manifold. Put

$$\mathcal{P}^1_H := \{L^1_1 \subset \mathbb{C}^4, \text{ linear subspace}, \dim L^1_1 = 1\} \simeq \mathbb{P}^3(\mathbb{C})$$
$$\mathcal{U}^1_H := \{L^1_2 \subset \mathbb{C}^4, \text{ linear subspace}, \dim L^1_2 = 2\} \simeq G(2, 4)$$

Then we have the following Penrose–Hurwitz correspondence:

$$\begin{array}{ccc} & \mathcal{I}_H & \\ {}^{\mu_1}\swarrow & & \searrow^{\nu_1} \\ \mathcal{P}^1_H & & \mathcal{U}^1_H \end{array}$$

Let $\pi_1\colon S_- \to \mathcal{U}^1_H$ be the universal vector bundle of $\mathcal{U}^1_H$, $\hat{\pi}_1$: det

$S_- \to \mathcal{U}^1_H$ be its determinant line bundle, and let $S_-^{\odot n}$ denote the nth symmetric tensor product of S_-. Let $M^{(j)}$ form the canonical coordinate covering of $\mathcal{U}^1_H$, constructed in Ławrynowicz and Suzuki (1998a, formula (19)). We use the van der Waerden spinor notation ∇ (Ławrynowicz and Suzuki, 1998a, formula (27)). A section Φ^1 on an open set U_1 of $\mathcal{U}^1_H$ of the vector bundle $S_-^{\odot n} \otimes \det S_-$,

$$\Phi^1 = \{\Phi_{\dot{A}\dot{B}\ldots\dot{D}}\} \in \Gamma(U_1, S_-^{\odot n} \otimes \det S_-)$$

is called a *spinor function of spin* $\frac{1}{2}n$ whenever the following *spinor equations* of spin $\frac{1}{2}n$ hold on $M_1^{(j)} \cap U_1$, $j = 1, 2, \ldots, 6$:

$$\nabla^{A\dot{A}}\Phi^1_{\dot{A}B\ldots\dot{D}} = 0, \qquad \text{where} \quad \dot{A}\dot{B}\ldots\dot{D} \text{ contains } n \text{ items} \tag{5}$$

Theorem A. (Ławrynowicz and Suzuki, 1998a). Hurwitz-twistor counterpart of Penrose's (1977) fundamental theorem, local version: Let $Z^{(n)}_{\mathcal{A}}(U)$ be the space of real-analytic solutions of the spinor equations (5) of spin $\frac{1}{2}n$ on an open set $U \subset \mathbb{C}^2$. Then they can be written as harmonic forms, i.e., there exists a one-to-one correspondence between spinor solutions and harmonic forms with respect to the (1, 1)-metric $ds^2 := dz^1\, d\bar{z}^1 - dz^2\, d\bar{z}^2$,

$$Z^{(n)}_{\mathcal{A}}(U) \simeq \mathbb{H}^1(U, \mathbb{C}^{2n-2})$$

$$\mathbb{H}^1(U, \mathbb{C}^{2n-2}) := \{\phi \in \Gamma^{1,0}(U, \mathbb{C}^{2n-2})\colon \partial\phi = 0 \text{ and } \vartheta\phi = 0\}$$

and ϑ is the formally adjoint operator of ∂ with respect to the indefinite fiber (2, 0)-metric $d\rho^2 := d\zeta^1\, d\bar{\zeta}^1 + d\zeta^2\, d\bar{\zeta}^2$.

Theorem B. (Ławrynowicz and Susuki, 1998a). Hurwitz-twistor counterpart of Penrose's (1977) fundamental theorem, semiglobal version: Let $Z^{(n)}_{\mathcal{H}}(U_1)$ be the space of holomorphic solutions of the spinor equations (5) of spin $\frac{1}{2}n$ on an open set U_1, whereas μ_1 and ν_1 are the related fiber bundles forming the Penrose-like diagram (4). We put $U_1' = \nu_1^{-1}(U_1)$ and $U_1'' = \mu_1 \circ \nu_1^{-1}(U_1)$. Then, if every fiber of μ_1 is connected, there exists a one-to-one correspondence

$$Z^{(n)}_{\mathcal{H}}(U_1) \simeq H^1(U_1'', \mathcal{O}(-n-2))$$

where H_1 denotes the one-dimensional Dolbeault cohomology group, $\mathcal{O}(-n-2) = \mathcal{O}([e]^{-n-2})$, and $[e]$ is the canonical effective divisor of $\mathbb{P}^3(\mathbb{C})$.

2. PSEUDOTWISTORS FOR HERMITIAN HURWITZ PAIRS

Let $\circ$: $\mathbb{R}^9(I_{8,1} \text{ or } I_{4,5}) \otimes_{\mathbb{R}} \mathbb{C}^{16}(I_{8,8}) \to \mathbb{C}^{16}(I_{8,8})$. An element $x \in \mathcal{A}_{8,1}$ is called a *pseudotwistor* and $x \in \mathcal{A}_{4,5}$ is called a *pseudobitwistor* whenever x has the form (2) and $\operatorname{im} x^2 = 0$. We denote the collection of pseudotwistors and pseudobitwistors by $\mathcal{T}_H = P_1$ and $\mathcal{T}_H = \mathrm{P}$, respectively; they coincide and are given by (3).

A relativistic *exciton* is a relativistic particle in the crystallographic lattice, situated at (x_1, x_2, x_3), whose previous position in that lattice, now a hole, was at (x_2, y_2, z_2) (Agranovich, 1968). The space-time elements are

$$\mathbb{R}^9(I_{7,2}): \qquad -ds^2 = \begin{cases} -c^2\,dt^2 + dx_1^2 + dy_1^2 + dz_1^2 + dx_2^2 + dy_2^2 + dz_2^2 - d\tau^2 + dX_{12}^2 \\ \begin{cases} -c^2\,dt^2 + dx_1^2 + dy_1^2 + dz_1^2 + dx_2^2 + dy_2^2 + dz_2^2 - d\tau^2 + dX_{12}^2 \\ \quad + dX_{21}^2, \qquad dz_2 = dz_1 = dz \end{cases} \end{cases}$$

$$\mathbb{R}^9(I_{1,8}): \qquad ds^2 = \begin{cases} c^2\,dt^2 - dx_1^2 - dy_1^2 - dz_1^2 - dx_2^2 - dy_2^2 - dz_2^2 - d\tau^2 - dX_{12}^2 \\ \begin{cases} c^2\,dt^2 - dx_1^2 - dy_1^2 - dz_1^2 - dx_2^2 - dy_2^2 - dz_2^2 - d\tau^2 - dX_{12}^2 \\ \quad - dX_{21}^2, \qquad dz_2 = dz_1 = dz \end{cases} \end{cases}$$

$$\mathbb{R}^9(I_{3,6}): \qquad ds^2 = \begin{cases} c^2\,dt^2 - dx_1^2 - dy_1^2 - dz_1^2 - dx_2^2 - dy_2^2 - dz_2^2 + d\tau^2 + dX_{12}^2 \\ \begin{cases} c^2\,dt^2 - dx_1^2 - dy_1^2 - dz_1^2 - dx_2^2 - dy_2^2 - dz_2^2 - d\tau^2 + dX_{12}^2 \\ \quad + dX_{21}^2, \quad dz_2 = dz_1 = dz \end{cases} \end{cases}$$

$$\mathbb{R}^9(I_{5,4}): \qquad -ds^2 = -c^2dt^2 + dx_1^2 + dy_1^2 + dz_1^2 + dx_2^2 + dy_2^2 + dz_2^2 - d\tau^2$$
$$-dX_{12}^2 - dX_{21}^2, \qquad dz_2 = dz_1 = dz$$

The terms $\pm dX_{12}^2$ or $-dX_{12}^2 - dX_{21}^2$ correspond to interactions. We associate the pseudotwistors with $\mathbb{R}^9(I_{1,8})$ and pseudobitwistors with $\mathbb{R}^9(I_{5,4})$. Solutions obtained for ds^2 indicate an interaction between the particle and the hole left by the particle. The interaction can be characterized in seven different ways corresponding to different types of fields: a spacelike and timelike field (with possible stochastic interpretation of $d\tau^2$), and giving a possibility of classifying fields from the point of view of their origin, e.g., electromagnetic, with a possibility of interpretation of their composition, e.g., $\mathbf{E}^2 + \mathbf{H}^2$, etc.

Lemma C. (Ławrynowicz and Suzuki, 1998). The expression (2) is an element of $\mathcal{J}_H = P_1$ or of $\mathcal{J}_H = \mathrm{P}$ if and only if the following $\binom{9}{4}$ equations hold;

$$\sum_{\alpha<\beta<\alpha'<\beta'} \operatorname{sgn}\begin{Bmatrix} a\ b\ a'\ b' \\ \alpha\ \beta\ \alpha'\ \beta' \end{Bmatrix} \xi_{\alpha\beta}\xi_{\alpha'\beta'} = 0,$$
$$a, b, \ldots, \alpha', \beta' \in \{1, 2, \ldots, 8\}$$

where sgn{ . . . } denotes the number of transpositions of two numbers of the sequence $(\alpha, \beta, \alpha', \beta')$ in order to obtain (a, b, a', b').

Proof:

$$\operatorname{im} x^2 = \sum_{a<b<a'<b'} \sum_{\alpha<\beta<\alpha'<\beta'} \operatorname{sgn}\begin{Bmatrix} a & b & a' & b' \\ \alpha & \beta & \alpha' & \beta' \end{Bmatrix} \xi_{\alpha\beta}\xi_{\alpha'\beta'}\, C_\alpha^{\#}\, C_\beta\, C_{\alpha'}\, C_{\#\beta'} \quad \blacksquare$$

Lemma D. $\mathcal{J}_H = P_1$ and $\mathcal{J}_H = \mathrm{P}$ admit the structure of a flag manifold:

$$\mathcal{J}_H = \left\{ (L_1^2, L_2^2): \begin{array}{c} L_1^2, L_2^2 \text{ are linear subspaces of } \mathbb{C}^8 \\ \\ L_1^2 \subset L_2^2, \dim L_1^2 = 1, \dim L_2^2 = 2 \end{array} \right\}$$

Hence $\mathcal{J}_H$ becomes a complex manifold. We have the following correspondences:

$$\mathcal{P}_H^2 := \{L_1^2 \subset \mathbb{C}^8\} \simeq \mathbb{P}^7(\mathbb{C}), \qquad \mathcal{U}_H^2 := \{L_2^2 \subset \mathbb{C}^8\} \simeq G(2,8)$$

$$\begin{array}{ccc} & P_1 & \\ {}^{\mu_2}\swarrow & & \searrow{}^{\nu_2} \\ \mathcal{P}_H^2 & & \mathcal{U}_H^2 \end{array} \quad \text{and} \quad \begin{array}{ccc} & P & \\ {}^{\mu_2}\swarrow & & \searrow{}^{\nu_2} \\ \mathcal{P}_H^2 & & \mathcal{U}_H^2 \end{array} \qquad (6)$$

Let $\pi_2: S_- \to \mathcal{U}_H^2$ be the universal vector bundle of $\mathcal{U}_H^2$, $\hat{\pi}_2: \det S_- \to \mathcal{U}_H^2$ its determinant line bundle, and $M^{(j)}$ form the canonical coordinate covering of $\mathcal{U}_H^2$. Consider a spinor function Φ^2 of spin $\frac{1}{2}n$ belonging to $\Gamma\,(U_2, S_-^{\odot n} \otimes \det S_-)$, U_2 being an open set of $\mathcal{U}_H^2$, and the corresponding spinor equations on $M_2^{(j)} \cap U_2$, $j = 1, 2, \ldots, 6$,

$$\nabla^{A\dot{A}}\Phi^2_{\dot{A}\dot{B}\ldots\dot{D}} = 0, \qquad \dot{A}\dot{B}\ldots\dot{D} \text{ contains } n \text{ items} \qquad (7)$$

Theorem C. (Ławrynowicz and Suzuki, 2000b). Pseudotwistor counterpart of Penrose's fundamental theorem, local version: Let $Z_{\mathcal{A}}^{(n)}(U)$ be the space of real-analytic solutions of the spinor equations (7) of spin $\frac{1}{2}n$ on an open set $U \subset \mathbb{C}^4$. Then they can be written as harmonic forms, i.e., there exists a one-to-one correspondence between spinor solutions and harmonic forms with respect to the metric

(0,4): $ds^2 := -\,dz^1\,d\bar{z}^1 - dz^2\,d\bar{z}^2 - dz^3\,d\bar{z}^3 - dz^4\,d\bar{z}^4$ pseudotwistors

(2,2): $ds^2 := dz^1\,d\bar{z}^1 + dz^2\,d\bar{z}^2 - dz^3\,d\bar{z}^3 - dz^4\,d\bar{z}^4$ pseudobitwistors

$$Z_{\mathcal{A}}^{(n)}(U) := \mathbb{H}^1(U, \mathbb{C}^{8n-8})$$

$$\mathbb{H}^1(U, \mathbb{C}^{8n-8}) := \{\phi \in \Gamma^{1,0}(U, \mathbb{C}^{8n-8}): \partial\phi = 0 \text{ and } \vartheta\phi = 0\}$$

and ϑ is the formally adjoint operator of ∂ with respect to the indefinite fiber (8,0)-metric

$$d\rho^2 := d\zeta^1\, d\bar{\zeta}^1 + d\zeta^2\, d\bar{\zeta}^2 + \cdots + d\zeta^8\, d\bar{\zeta}^8$$

Theorem D. (Ławrynowicz and Suzuki, 2000b). Pseudotwistor counterpart of Penrose's fundamental theorem, semiglobal version: Let $Z^{(n)}_{\mathscr{H}}\,(U_2)$ be the space of holomorphic solutions of the spinor equations (7) of spin $\frac{1}{2}n$ on an open set U_2, with μ_2 and ν_2 the related fiber bundles forming the Penrose-like diagrams (6). We put $U_2' = \nu_2^{-1}\,(U_2)$ and $U_2'' = \mu_2 \circ \nu_2^{-1}\,(U_2)$. Then, if every fiber of μ_2 is connoted, there exists a one-to-one correspondence

$$Z^{(n)}_{\mathscr{H}}\,(U_2) \simeq H^1\,(U_2'', \mathbb{O}(-\alpha n - \beta))$$

where α and β, $\beta \geq 2$, are some positive integers.

3. COMPLEX STRUCTURES OF SPINORS

For spinor equations, we refer to Lounesto (1997, especially p. viii) and Ławrynowicz and Suzuki (2000a). We consider complex structures of spinors, more exactly, to decide whether the following isometric embedding becomes the part of a holomorphic mapping:

$$\mathbb{C}^2 \simeq \mathbb{R}^4 \ni \mathbf{x} \overset{\iota}{\mapsto} \sum_{\alpha=1}^{3} x^\alpha\, S_\alpha + x^4 I_4 \in G(2, 4) \tag{8}$$

$$\mathbb{C}^4 \simeq \mathbb{R}^8 \ni \mathrm{x} \overset{\iota}{\mapsto} \sum_{\alpha=1}^{7} x^\alpha\, S_\alpha + x^8 I_8 \in G(8, 16) \tag{9}$$

Lemma 1. The isometric embeddings ι of the Hurwitz algebras $\mathscr{H}_{4,0}$, $\mathscr{H}_{2,2}$, $\mathscr{H}_{0,4}$ admit complex structures. The Weyl equations and the isometric embedding ι are as follows:

$$\mathscr{H}_{4,0}:\quad [i\sigma_1\partial^1 + i\sigma_2\partial^2 + i\sigma_3\partial^3 + I_4\partial^4]\mathbf{f}_+ = 0, \qquad \iota(u, v) = A_1 := \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$$

$$u = x^4 + ix^3, \qquad v = x^2 + ix^1$$

$$\mathscr{H}_{2,2}:\quad [\sigma_1\partial^1 - \sigma_2\partial^2 + i\sigma_3\partial^3 + I_4\partial^4]\mathbf{f}_+ = 0, \qquad \iota(u, v) = A_2 := \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}$$

$$u = x^4 + ix^3, \qquad v = x^1 + ix^2$$

$$\mathscr{H}_{0,4}:\quad [i\sigma_1\partial^1 - \sigma_2\partial^2 + i\sigma_3\partial^3 + I_2\partial^4]\mathbf{f}_+ = 0,$$

$$\iota(u, v) = A_1, \quad u = x^4 + ix^3, \quad v = x^2 + ix^1$$

Lemma 2. The isometric embeddings given in Lemma 1 are real parts of holomorphic mappings.

Proof. Using in Lemma 1 the following notation and the complex conjugates

$$T_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \overline{T}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \overline{T}_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$\iota(u, v) = \iota_H \oplus \bar{\iota}_H, \qquad \iota_H = uT_1 + vT_2$$

we can realize the isometric mapping as the real form of a holomorphic embedding for $\mathcal{H}_{2,2}$. The other cases are similar. ■

The purpose of this section is the treatment of $\mathcal{H}_{\sigma,s}$, $\sigma + s = 8$.

Atomization Theorem. There exist complex structures on isometric embeddings for Hermitian Hurwitz pairs related to the algebras $\mathcal{H}_{0,8}$, $\mathcal{H}_{2,6}$, $\mathcal{H}_{4,4}$, $\mathcal{H}_{6,2}$, and $\mathcal{H}_{8,0}$ so that the embeddings are real parts of holomorphic mappings.

Proof. The proof is given by making the list of complex structures. ■

In the sequel,

$$\mathcal{T}_1 = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \overline{\mathcal{T}}_1 = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix},$$

$$\mathcal{T}_2 = \begin{pmatrix} 0 & I_2 \\ 0 & 0 \end{pmatrix}, \qquad \overline{\mathcal{T}}_2 = \begin{pmatrix} 0 & 0 \\ -I_2 & 0 \end{pmatrix}$$

For $\mathcal{H}_{0,8}$, the embedding is (Ławrynowicz and Suzuki, 2000a, Section 2.1.1)

$$u = x^3 + ix^8, \qquad v = x^1 + ix^2, \qquad w = x^4 + ix^5, \qquad t = x^6 + ix^7$$

$$\iota(u, v, w, t) = \begin{pmatrix} A_3 & A_5 & A_7 & 0 \\ A_6 & A_4 & 0 & A_7 \\ A_8 & 0 & A_4 & -A_5 \\ 0 & A_8 & -A_6 & A_3 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} \bar{u} & \bar{v} \\ v & -u \end{pmatrix},$$

$$A_4 = \begin{pmatrix} -u & -\bar{v} \\ -v & \bar{u} \end{pmatrix}, \qquad A_5 = \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}, \qquad A_6 = \begin{pmatrix} \bar{w} & 0 \\ 0 & \bar{w} \end{pmatrix},$$

$$A_7 = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, \qquad A_8 = \begin{pmatrix} \bar{t} & 0 \\ 0 & \bar{t} \end{pmatrix}$$

The holomorphic mapping is

$$\iota_H = u\mathbb{T}_1 + v\mathbb{T}_2 + t\mathbb{T}_3 + w\mathbb{T}_4 \tag{10}$$

$$\mathbb{T}_1 = \mathrm{diag}(-\overline{T}_1, -T_1, -T_1, -\overline{T}_1), \qquad \mathbb{T}_2 = \mathrm{diag}(-\overline{T}_2, -\overline{T}_2, -\overline{T}_2, -\overline{T}_2)$$

$$\overline{\mathbb{T}}_1 = \mathrm{diag}(T_1, \overline{T}_1, \overline{T}_1, T_1), \qquad \overline{\mathbb{T}}_2 = \mathrm{diag}(T_2, T_2, T_2, T_2)$$

$$\mathbb{T}_3 = \begin{pmatrix} 0 & I_4 \\ 0 & 0 \end{pmatrix}, \qquad \mathbb{T}_4 = \begin{pmatrix} \mathcal{T}_2 & 0 \\ 0 & -\mathcal{T}_2 \end{pmatrix},$$

$$\overline{\mathbb{T}}_3 = \begin{pmatrix} 0 & 0 \\ I_4 & 0 \end{pmatrix}, \qquad \overline{\mathbb{T}}_4 = \begin{pmatrix} -\overline{\mathcal{T}}_2 & 0 \\ 0 & \overline{\mathcal{T}}_2 \end{pmatrix}$$

and the isometric embedding satisfies the formula

$$\iota = \iota_H \oplus \bar{\iota}_H \tag{11}$$

For $\mathcal{H}_{6,2}$, the embedding is (Ławrynowicz and Suzuki, 2000a, Section 2.2.1)

$$\iota(v, w, t, x) = \begin{pmatrix} A_9 & 0 & A_5 & A_1 \\ 0 & -A_{10} & A_{11} & A_6 \\ -A_6 & A_1 & -A_{10} & 0 \\ A_{11} & -A_5 & 0 & A_9 \end{pmatrix}$$

$$u = x^4 + ix^3, \qquad v = x^2 + ix^1, \qquad w = x^5 + ix^6, \qquad t = x^7 + ix^8$$

$$A_9 = \begin{pmatrix} t & 0 \\ 0 & -\bar{t} \end{pmatrix}, \qquad A_{10} = \begin{pmatrix} \bar{t} & 0 \\ 0 & -t \end{pmatrix}, \qquad A_{11} = \begin{pmatrix} -\bar{u} & v \\ -\bar{v} & -u \end{pmatrix}$$

The matrices $A_1, A_2, \ldots, A_{11}$ will be called *atoms*.

The holomorphic mapping is given by (10), where

$$\mathbb{T}_1 = \mathrm{diag}(T_1, \overline{T}_1, \overline{T}_1, T_1), \qquad \mathbb{T}_2 = \begin{pmatrix} 0 & \mathcal{T}_1 \\ \overline{\mathcal{T}}_1 & 0 \end{pmatrix}$$

$$\mathbb{T}_3 = \mathrm{diag}^*(-\overline{T}_1, T_1, -\overline{T}_1, T_1), \qquad \mathbb{T}_4 = \mathrm{diag}^*(T_2, T_2, T_2, T_2)$$

$$\overline{\mathbb{T}}_1 = \mathrm{diag}(-\overline{T}_1, -T_1, -T_1, -\overline{T}_1), \qquad \overline{\mathbb{T}}_2 = \begin{pmatrix} 0 & \overline{\mathcal{T}}_1 \\ -\mathcal{T}_1 & 0 \end{pmatrix}$$

$$\overline{\mathbb{T}}_3 = \mathrm{diag}^*(-T_1, \overline{T}_1, -T_1, \overline{T}_1), \qquad \overline{\mathbb{T}}_4 = \mathrm{diag}^*(\overline{T}_2, \overline{T}_2, \overline{T}_2, \overline{T}_2)$$

and the isometric embedding satisfies the formula (11).

For $\mathcal{H}_{2,6}$, the embedding is (Ławrynowicz and Suzuki, 2000a, Section 2.3.1)

$$\iota(v, w, t, x) = \begin{pmatrix} A_3 & A_7 & A_5 & 0 \\ A_8 & A_4 & 0 & A_5 \\ -A_6 & 0 & A_3 & -A_7 \\ 0 & -A_6 & -A_8 & A_4 \end{pmatrix}$$

$$u = x^3 + ix^8, \qquad v = x^1 + ix^2, \qquad w = x^7 + ix^6 \qquad t = x^4 + ix^5$$

The holomorphic mapping is as given by (10), where

$$\mathbb{T}_1 = \operatorname{diag}(-\overline{T}_1, -T_1, -\overline{T}_1, -T_1), \qquad \mathbb{T}_2 = \operatorname{diag}(-\overline{T}_2, \overline{T}_2, -\overline{T}_2, \overline{T}_2)$$

$$\overline{\mathbb{T}}_1 = \operatorname{diag}(-\overline{T}_1, -T_1, -T_1, -\overline{T}_1), \qquad \overline{\mathbb{T}}_2 = \operatorname{diag}(T_2, -T_1, T_2, -T_1)$$

$$\mathbb{T}_3 = \begin{pmatrix} \mathcal{T}_2 & 0 \\ 0 & -\mathcal{T}_2 \end{pmatrix}, \qquad \mathbb{T}_4 = \begin{pmatrix} 0 & I_4 \\ 0 & 0 \end{pmatrix},$$

$$\overline{\mathbb{T}}_3 = \begin{pmatrix} -\overline{\mathcal{T}}_2 & 0 \\ 0 & \overline{\mathcal{T}}_2 \end{pmatrix}, \qquad \overline{\mathbb{T}}_4 = \begin{pmatrix} 0 & 0 \\ -I_4 & 0 \end{pmatrix}$$

and the isometric embedding satisfies the formula (11).

For $\mathcal{H}_{4,4}$, the embedding is (Ławrynowicz and Suzuki, 2000a, Section 2.4.1)

$$\iota(u, v, w, t) = \begin{pmatrix} A_3 & 0 & A_5 & A_7 \\ 0 & A_3 & -A_8 & A_6 \\ -A_6 & A_7 & A_4 & 0 \\ -A_8 & -A_5 & 0 & A_4 \end{pmatrix}$$

$$u = x^3 + ix^8, \qquad v = x^1 + ix^2, \qquad w = x^6 + ix^7, \qquad t = x^4 + ix^5$$

The holomorphic mapping is as given by (10), where

$$\mathbb{T}_1 = \operatorname{diag}(-\overline{T}_1, -\overline{T}_1, -T_1, -T_1), \qquad \mathbb{T}_2 = \operatorname{diag}(\overline{T}_2, \overline{T}_2, \overline{T}_2, \overline{T}_2)$$

$$\overline{\mathbb{T}}_1 = \operatorname{diag}(T_1, T_1, \overline{T}_1, \overline{T}_1) \qquad , \qquad \overline{\mathbb{T}}_2 = \operatorname{diag}(T_2, T_2, -T_2, -T_2)$$

$$\mathbb{T}_3 = \begin{pmatrix} 0 & \mathcal{T}_2 \\ \mathcal{T}_2 & 0 \end{pmatrix}, \qquad \mathbb{T}_4 = \begin{pmatrix} 0 & \mathcal{T}_1 \\ -\overline{\mathcal{T}}_1 & 0 \end{pmatrix},$$

$$\overline{\mathbb{T}}_3 = \begin{pmatrix} 0 & -\overline{\mathcal{T}}_2 \\ -\overline{\mathcal{T}}_2 & 0 \end{pmatrix}, \qquad \overline{\mathbb{T}}_4 = \begin{pmatrix} 0 & \overline{\mathcal{T}}_1 \\ -\mathcal{T}_1 & 0 \end{pmatrix}$$

and the isometric embedding satisfies the formula (11).

REFERENCES

Agranovich, Vladimir M. (1968). *Teoriya èksitonov*, Nauka, Moscow.

Ławrynowicz, Julian, and Jakub Rembieliński (1985). Pseudo-Euclidean Hurwitz pairs and generalized Fueter equations, Institute of Mathematics of the Polish Academy of Sciences Preprint No. 355.

Ławrynowicz, Julian, and Jakub Rembieliński (1986a). Pseudo-Euclidean Hurwitz pairs and generalized Fueter equations, in: J. S. R. Chisholm and A. K. Common, Editors, *Clifford Algebras and Their Applications in Mathematical Physics*, Reidel, Dordrecht, pp. 39–48.

Ławrynowicz, Julian, and Jakub Rembieliński (1986b). Pseudo-Euclidean Hurwitz pairs and the Kaluża–Klein theories, Institute of Physics of the University of Łódź Preprint No. 86-8.

Ławrynowicz, Julian, and Jakub Rembieliński (1987). Pseudo-Euclidean Hurwitz pairs and the Kaluża–Klein theories, *Journal of Physics A: Mathematical and General*, **20**, 5831–5848.

Ławrynowicz, Julian, and Osamu Suzuki (1998a). The twistor theory of the Hermitian Hurwitz pair $(\mathbb{C}^4(I_{2,2}), \mathbb{R}^5(I_{2,3}))$ *Advances in Applied Clifford Algebras*, **8**, 147–179 [Institute of Mathematics of the Polish Academy of Sciences Preprint No. 569 (1997)].

Ławrynowicz, Julian, and Osamu Suzuki (1998b). An approach to the 5-, 9-, and 13-dimensional complex dynamics I. Dynamical aspects, *Bulletin de la Société des Sciences et des Lettres de Lódź 48, Série: Recherches sur les Déformations*, **25**, 7–39.

Ławrynowicz, Julian, and Osamu Suzuki (2000a). An introduction to pseudotwistors—Spinor equations, *Revue Roumaine des Mathématiques Pures et Appliquées*.

Ławrynowicz, Julian, and Osamu Suzuki (2000b). An introduction to pseudotwistors—Spinor solutions versus harmonic forms and cohomology groups, in: Rafał Abłamowics and Bertfried Fauser, Editors, *Clifford Algebras and Their Applications in Mathematical Physics, Volume 1, Algebra and Physics*, Birkhäuser Basel.

Lounesto, Pertti (1997). *Clifford Algebras and Spinors*, Cambridge University Press, Cambridge.

Penrose, Roger (1977). The twistor programme, *Reports on Mathematical Physics*, **12**, 65–76.

Wells, Raymond, O, Jr (1982). Complex geometry in mathematical physics, Séminaire de Mathématiques supérieures, Université de Montréal, Montreal.